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Solar Sail Parking in Restricted Three-Body Systems

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Stationary solutions to the restricted three-body problem for solar sail spacecraft in the Earth-sun and Earth-moon systems are investigated. It is found that the usual five Lagrange points are extended to a continuum of new artificial points that form level surfaces parameterized by the sail mass per unit area. Analytic expressions for the sail mass per unit area and the sail attitude required for these stationary solutions are obtained and the stability of the solutions examined. It is found that although in general the solutions are unstable, a simple closed-loop control scheme may be developed to ensure asymptotic stability.

Nomenclature

A	= sail area
a, a_j	= radiation pressure acceleration, characteristic polynomial coefficients
C	= controllability matrix
I	= identity matrix
L	= Lagrange point
$M_{1,2,3}$, M^*	= matrices of Eq. (16), $M_2 - M_1$
m, m_p	= spacecraft mass, payload mass
N	= first-order change in radiation pressure acceleration, $\partial a/\partial n$
n	= unit vector normal to sail surface
P	= sixth-order characteristic polynomial
P	= system matrix
Q	= input distribution matrix
r	= Cartesian position vector (x, y, z)
S, $S_{1,2}$	= surfaces bounding regions of existence of stationary solutions
S	= Sun-line direction in Earth-moon system
s	= complex variable, $\gamma + i\omega$
T	= surfaces bounding regions of existence of stationary solutions
U	= corotating three-body potential
x	= state vector
β	= dimensionless sail loading parameter, σ^*/σ
γ	= Re(s)
δ	= perturbation of sail position vector, $r \rightarrow r + \delta$
λ	= scalar feedback gains
$\Lambda_{1,2}$	= feedback gain matrices to sail attitude
μ	= system mass ratio, $m_2/(m_1 + m_2)$
σ, σ^*	= sail mass per unit area, 1.53 gm^{-2}
Φ	= three-body gravitational potential
Ψ	= centrifugal potential
ω	= Im(s)
Ω, Ω_*	= corotating frame angular velocity, angular rate of sun line
∇	= gradient operator
$\hat{( )}$	= unit vector

Subscripts

0	= initial value
1	= with respect to larger primary mass
2	= with respect to smaller primary mass

Introduction

IN this paper stationary solutions for solar sail spacecraft in the Earth-sun and Earth-moon restricted three-body dynamical systems are investigated. The classical circular restricted three-body problem has five well-known stationary solutions, the so-called Lagrange points L_j (j = 1, 5), where an infinitesimal mass will remain at rest with respect to the two primary masses of the system. It is found that in general the collinear points L_j (j = 1, 3) are unstable, whereas the triangular points L_j (j = 4, 5) are stable in the Lyapunov sense.[1] The classical restricted three-body problem has been extended to include a radiation pressure force from either or both of the primary masses, exerted on the infinitesimal mass.[2] This formulation generates four new additional stationary solutions with interesting stability characteristics. The radiation pressure force vector is, however, constrained to lie along the primary mass-infinitesimal mass line.

For the Earth-sun-sail three-body system the sail attitude may be freely oriented so that the solar radiation pressure force vector is not constrained to lie along the sun-sail line. Furthermore, the magnitude of the solar radiation pressure force may be chosen through the total spacecraft mass per unit area. Therefore, since certain parameters of the system can be arbitrarily chosen, it is clear that rich new possibilities for artificial stationary solutions will arise. In fact, it will be demonstrated that there is a continuum of new stationary solutions parameterized by the sail attitude and sail mass per unit area. The dynamical stability of these new stationary solutions is investigated and their instability established. Therefore, a simple closed-loop control scheme is developed whereby a feedback to the sail attitude is used to ensure the asymptotic stability of the solutions. These stationary solutions are an extension to previously developed heliocentric halo-type orbits for solar sail spacecraft.[3,4]

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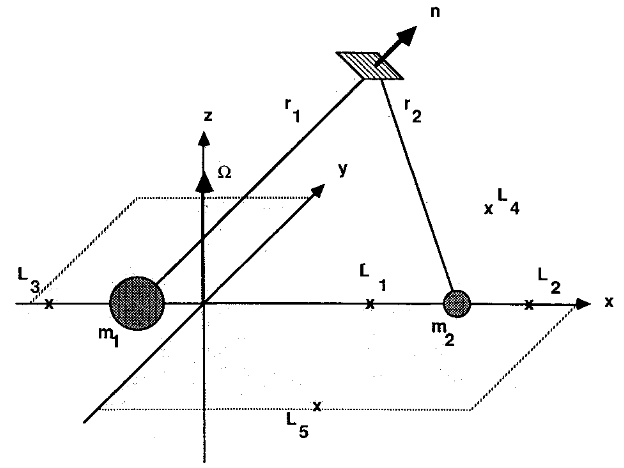

Fig. 1 Schematic geometry of the Earth-sun solar sail restricted three-body system.

For the Earth-moon three-body system the dynamics are no longer autonomous as the sun line rotates once per synodic month with respect to the Earth-moon corotating reference frame. However, short duration stationary solutions are again possible by utilizing small trims in the sail area to compensate for the rotation of the sun line. Using this technique the spacecraft may be parked for a short duration in the Earth-moon system. These new three-body stationary solutions have potential applications for space science missions and for the parking of solar sail spacecraft on interplanetary trajectories.

In this initial, first-order analysis, the orbit of the two primary masses will be taken to be exactly circular, and no other gravitational accelerations other than those due to the primary masses will be considered. These idealizations allow the essential results to be obtained without undue complexity. In practice, however, these effects will have bearing on stability and control and should be investigated in future studies. Similarly, it will be assumed that the sail is perfectly reflecting. The nonperfect reflectivity of a real sail will cause small modifications to the results obtained, but will not change their fundamental character.

Dynamical Equations for the Earth-Sun System

Consider now an idealized, perfectly reflecting, planar solar sail in a corotating Cartesian reference frame of angular velocity $\boldsymbol{\Omega}$ with two point primary masses m_1 and m_2, as shown in Fig. 1. The sail attitude is defined by $\boldsymbol{n}$ and is fixed in the corotating frame. Since the sail attitude is to be fixed in the corotating frame, the sail must rotate about the normal to the plane of the system once per year with respect to a fixed inertial frame. The ratio of the solar radiation pressure force to the solar gravitational force exerted on the sail is given by dimensionless sail loading parameter $\beta = \sigma^*/\sigma$. By considering the ratio of forces, it may be shown that the critical mass per unit area $\sigma^* = 1.53\ \mathrm{gm}^{-2}$. The units of the system will be chosen such that the gravitational constant, the distance between the two primary masses, the sum of the primary masses, and so the angular velocity of corotation are all taken to be unity.

The vector dynamical equation for a solar sail in the corotating frame may be written as

$$\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} + 2\boldsymbol{\Omega} \times \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) = \boldsymbol{a} - \nabla\Phi_3(\boldsymbol{r}) \tag{1}$$

where the three-body gravitational potential $\Phi_3(\boldsymbol{r})$ and the nonconservative solar radiation pressure acceleration $\boldsymbol{a}$ are given by

$$\Phi_3(\boldsymbol{r}) = -\left\{\frac{1-\mu}{|\boldsymbol{r}_1|} + \frac{\mu}{|\boldsymbol{r}_2|}\right\}, \qquad \boldsymbol{a} = \beta\,\frac{1-\mu}{|\boldsymbol{r}_1|^2}\,(\hat{\boldsymbol{r}}_1\cdot\boldsymbol{n})^2\boldsymbol{n} \tag{2}$$

Although the system is nonconservative, it is, however, autonomous owing to the corotation of the reference frame. The solar radiation pressure force vector can never be directed sunward so that the sail attitude is constrained such that $\boldsymbol{r}_1\cdot\boldsymbol{n} \geq 0$. The sail position vectors are defined with respect to the Cartesian frame by

$$\boldsymbol{r}_1 = (x+\mu, y, z), \qquad \boldsymbol{r}_2 = (x-(1-\mu), y, z) \tag{3}$$

Since the centrifugal term in Eq. (1) is conservative it may be written as a scalar potential $\Psi(\boldsymbol{r})$ such that

$$\nabla\Psi(\boldsymbol{r}) = \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r}), \qquad \Psi(\boldsymbol{r}) = -\tfrac{1}{2}|\boldsymbol{\Omega}\times\boldsymbol{r}|^2 \tag{4}$$

Defining a new corotating potential $U(\boldsymbol{r}) = \Phi_3(\boldsymbol{r}) + \Psi(\boldsymbol{r})$, a reduced dynamical equation is obtained.

$$\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} + 2\boldsymbol{\Omega}\times\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} + \nabla U(\boldsymbol{r}) = \boldsymbol{a} \tag{5}$$

In the corotating frame stationary solutions are required so that the first two terms of Eq. (5) vanish. The five classical Lagrange solutions $\boldsymbol{r}^{L_j}$ ($j = 1, 5$) are then given by the solutions to the equation $\nabla U(\boldsymbol{r}) = 0$. However, for the solar sail three-body system there exists an additional acceleration term $\boldsymbol{a}$ that is a function of the sail loading parameter β and the sail attitude $\boldsymbol{n}$ so that new artificial stationary solutions may be generated.

Since the vector $\boldsymbol{a}$ is oriented in direction $\boldsymbol{n}$, taking the vector product of $\boldsymbol{n}$ with Eq. (5) it follows that

$$\nabla U(\boldsymbol{r})\times\boldsymbol{n} = 0 \Rightarrow \boldsymbol{n} = \lambda\nabla U(\boldsymbol{r}) \tag{6}$$

where λ is an arbitrary scalar multiplier. Using the normalization condition $|\boldsymbol{n}| = 1$, λ is identified as $|\nabla U(\boldsymbol{r})|^{-1}$ so that the sail attitude required for a stationary solution is given by

$$\boldsymbol{n} = \frac{\nabla U(\boldsymbol{r})}{|\nabla U(\boldsymbol{r})|} \tag{7}$$

The sail attitude may then be expressed in terms of two angles (α, χ), defined with respect to the coordinate triad $[\hat{\boldsymbol{r}}_1, \hat{\boldsymbol{r}}_1\times\boldsymbol{\Omega}, (\hat{\boldsymbol{r}}_1\times\boldsymbol{\Omega})\times\hat{\boldsymbol{r}}_1]$ centered on the sail. The pitch angle α is defined as the angle of $\boldsymbol{n}$ with respect to $\hat{\boldsymbol{r}}_1$, and the clock angle χ is defined as the angle of the projection of $\boldsymbol{n}$ in the plane normal to $\hat{\boldsymbol{r}}_1$ with respect to $(\hat{\boldsymbol{r}}_1\times\boldsymbol{\Omega})$. Therefore, taking vector and scalar products of Eq. (7) with $\hat{\boldsymbol{r}}_1$, these angles may be written as

$$\tan\alpha(\boldsymbol{r}) = \frac{|\boldsymbol{r}_1\times\nabla U(\boldsymbol{r})|}{\boldsymbol{r}_1\cdot\nabla U(\boldsymbol{r})} \tag{8a}$$

$$\tan\chi(\boldsymbol{r}) = \frac{|(\boldsymbol{r}_1\times\boldsymbol{\Omega})\times[\boldsymbol{r}_1\times\nabla U(\boldsymbol{r})]|}{(\boldsymbol{r}_1\times\boldsymbol{\Omega})\cdot[\boldsymbol{r}_1\times\nabla U(\boldsymbol{r})]} \tag{8b}$$

The sail loading required may also be obtained by taking a scalar product of Eq. (5) with $\boldsymbol{n}$. Requiring a stationary solution, it is found that

$$\beta(\boldsymbol{r}) = (1-\mu)^{-1}|\boldsymbol{r}_1|^2\,\frac{\nabla U(\boldsymbol{r})\cdot\boldsymbol{n}}{(\hat{\boldsymbol{r}}_1\cdot\boldsymbol{n})^2} \tag{9}$$

Therefore, general vector valued functions for the sail attitude and loading required for stationary solutions have been obtained in terms of the corotating three-body potential $U(\boldsymbol{r})$. Since the sail loading and attitude may be chosen at will, the set of five Lagrange stationary solutions will be replaced by an infinite set of artificially generated stationary solutions. The classical solutions then correspond to the subset $\beta = 0$. This infinite set of solutions is parameterized into level surfaces by the sail loading β. A particular stationary solution on a level surface is then defined by the two attitude angles (α, χ).

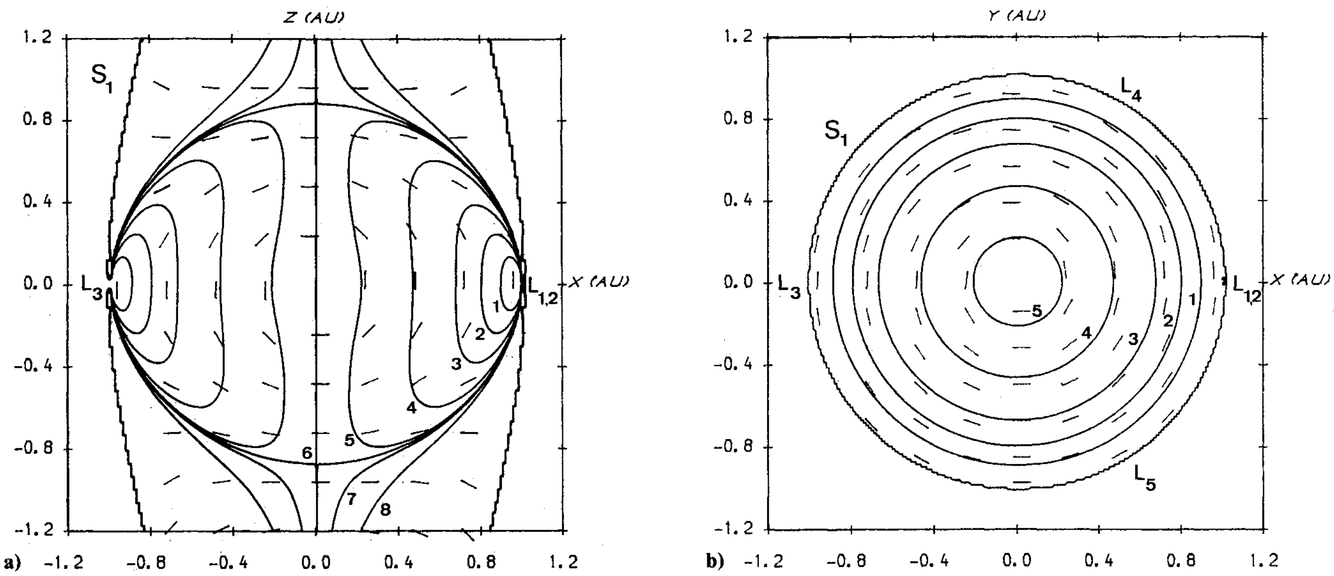

Fig. 2 Section of the level surfaces in the Earth-sun system a) normal to the plane of the system and b) in the plane of the system. Required sail loadings are given by: 1) 0.3, 2) 0.5, 3) 0.7, 4) 0.9, 5) 1.0, 6) 1.01, and 7) 1.1.

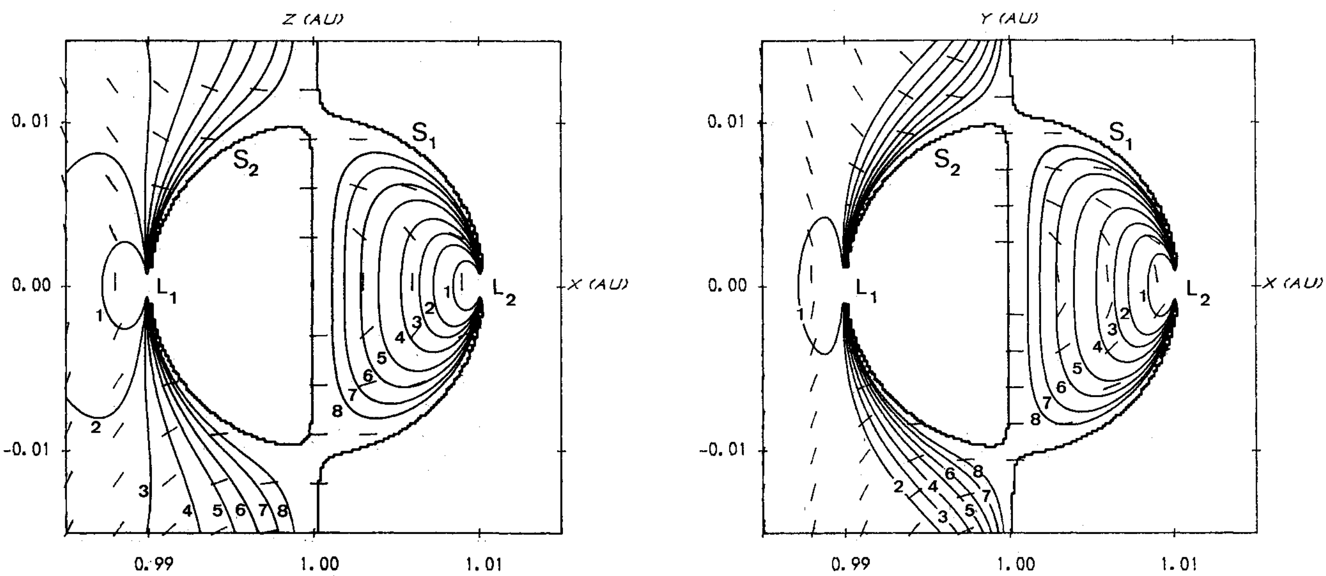

Fig. 3 Section of the level surfaces in the near Earth region a) normal to the plane of the system and b) in the plane of the system. Required sail loadings are given by: 1) 0.02, 2) 0.04, 3) 0.06, 4) 0.1, 5) 0.2, 6) 0.4, 7) 1.0, and 8) 3.0.

Existence of Stationary Solutions

Now that the existence of new stationary solutions has been established, the regions in which these solutions may exist must be investigated. These regions are defined by the constraint

$$r_1 \cdot \nabla U(r) \geq 0 \tag{10}$$

with the boundary surface defined by an equality in Eq. (10). This constraint may be understood physically since the solar radiation pressure acceleration vector $\boldsymbol{a}$, and so the sail attitude vector $\boldsymbol{n}$, can never be directed sunward. The sail pitch angle is therefore constrained such that $|\alpha| < \pi/2$.

In scalar form the three-body potential $U(r) = \Phi_3(r) + \Psi(r)$ may be written as

$$U(r) = -\left[\frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}\right] \tag{11}$$

Therefore, evaluating the gradient of $U(r)$ in Eq. (10) a function $S(r) = 0$ is obtained

$$S(r) = x(x+\mu) + y^2 - \frac{1-\mu}{r_1} - \frac{r_1 \cdot r_2}{r_2^3} \tag{12}$$

The function $S(r)$ has two topologically disconnected boundary surfaces S_1 and S_2 that define the boundary to the region of existence of stationary solutions (see Figs. 2 and 3). The region of existence of the stationary solutions lies between these two surfaces.

The outer surface S_1 possesses a cylindrical topology and excludes solutions along the x axis from $-\infty < x < x^{L_3}$ and $x^{L_2} < x < +\infty$, whereas the inner surface S_2 excludes solutions along $x^{L_1} < x < 1-\mu$. All of the five classical Lagrange solutions lie on S_1 and S_2 since they are the solutions $\nabla U(r) = 0$ of Eq. (10). In general the sail loading level surfaces approach the boundary surface asymptotically with $\beta \to \infty$, as then $r_1 \cdot \nabla U(r) \to 0$ in Eq. (9).

Stationary Solutions in the Earth-Sun System

Level surfaces of constant sail loading may now be generated from Eq. (9) for the Earth-sun system ($\mu = 3.036 \times 10^{-6}$). For ease of illustration, sections of the surfaces through the x-y and x-z planes are used. Then, only the pitch angle α will be required to completely describe the sail attitude required for a stationary solution. In general, though, the two angles (α, χ) are required to describe the sail attitude for a stationary solution at some arbitrary position.

The sections of the level surfaces generated by Eq. (9) are shown in Figs. 2 and 3. The sections define families of one parameter level curves representing subsets of the continuum of new artificial stationary solutions with equal sail loading. The sail attitude required is also shown. From Fig. 2 it can be seen that in the far Earth region the level surfaces are a family of topologically nested tori with the inner radius of the torus vanishing as $\beta \to 1$. In the plane of the system the level curves are near circular with the sail pitch angle $\alpha \simeq 0$. These curves represent the sail loading required for a circular heliocentric orbit with an orbital period of 1 yr. Out of the plane of the system the solutions are essentially Earth synchronous heliocentric halo-type orbits.[3,4] Along the z axis of the system there are stationary solutions above the poles of the sun with $\beta = 1$.

A detailed plot of the sections of the level surfaces near the Earth is shown in Fig. 3. It can be seen that the level surfaces around L_1 and L_2 accessible to the solar sail expand with increased sail loading, but always contain the classical Lagrange point they are associated with, as this corresponds to the solution $\beta \neq 0$, $r_1 \cdot n = 0$.

For $\beta \geq 1$ there are no solutions in the plane of the system, except at the five classical Lagrange points where $r_1 \cdot n = 0$. It can be seen in Fig. 2 that away from the Earth the level surfaces have undergone a topological change for $\beta > 1$ and have become a family of nested cylinders. There are also no intersections with, and so no solutions in, the x-y plane except at the $L_{4,5}$ points where $r_1 \cdot n = 0$ and the sail loading is undefined. There are, however, out-of-plane solutions corresponding to halo-type orbits at greater distances above the plane of the system. In the near Earth region, Fig. 3, it is seen that for an increased sail loading the level surfaces continue to expand, asymptotically approaching the boundary surfaces S_1 and S_2.

Stability and Control

Now that the existence of the artificial stationary solutions has been established, it is necessary to examine their stability. The general vector dynamical equation is given by Eq. (5) as

$$\frac{d^2 r}{dt^2} + 2\Omega \times \frac{dr}{dt} + \nabla U(r) = a \tag{13}$$

It will be assumed that the sail is stationary on some level surface at a point r_0. Then the dynamical equation in a local neighborhood of r_0 is obtained in the usual manner using an arbitrary perturbation δ, such that $r_0 \to r_0 + \delta$. Since r_0 is a stationary solution, a variational equation is obtained.

$$\frac{d^2 \delta}{dt^2} + 2\Omega \times \frac{d\delta}{dt} + \nabla U(r_0 + \delta) - a(r_0 + \delta) = 0 \tag{14}$$

The potential gradient and the solar radiation pressure acceleration may be expanded in trivariate Taylor series about the stationary solution to first order as

$$\nabla U(r_0 + \delta) = \nabla U(r_0) + \frac{\partial}{\partial r} \nabla U(r) \bigg|_{r = r_0} \delta + 0(|\delta|^2) \tag{15a}$$

$$a(r_0 + \delta) = a(r_0) + \frac{\partial}{\partial r} a(r) \bigg|_{r = r_0, n = n_0} \delta + 0(|\delta|^2) \tag{15b}$$

Then, since $\nabla U(r_0) = a(r_0)$ for a stationary solution, a linear variational equation is obtained

$$\frac{d^2 \delta}{dt^2} + M_1 \frac{d\delta}{dt} + (M_2 - M_3)\delta = 0 \tag{16}$$

where $M_{2,3}$, the gravity and radiation gradient tensors, and the skew symmetric gyroscopic matrix M_1 are given by

$$M_1 = \begin{Bmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{Bmatrix}, \qquad M_2 = \{U_{ij}\}, \qquad M_3 = \{a_{ij}\} \qquad (i, j) \in (x, y, z) \tag{17}$$

where U_{ij} is the (i, j) partial derivative of the potential with respect to the Cartesian axes and a_{ij} is the jth derivative of the ith component of the solar radiation pressure acceleration. The stability of the system may be investigated in the usual manner by examining the system eigenvalues resulting from the characteristic polynomial. This investigation may be carried out by substituting an exponential solution of the form

$$\delta = \delta_0 e^{st}, \qquad s = \gamma + i\omega, \qquad i = \sqrt{-1} \tag{18}$$

Substituting this solution into Eq. (16) yields a matrix equation of the form

$$(s^2 I + sM_1 + M^*)\delta_0 = 0 \tag{19}$$

where $M^* = M_2 - M_1$. For nontrivial solutions a vanishing secular determinant is required, which then gives the characteristic polynomial of the system $P(s) = 0$

$$P(s) = \sum_{j=0}^{6} a_{6-j} s^j \tag{20}$$

where, owing to the fundamental theorem of algebra, $P(s)$ has six roots $s_j = \gamma_j + i\omega_j (j = 1, 6)$. For asymptotic stability it is required that all of the system eigenvalues are in the left-hand complex plane so that $\gamma_j < 0$ $(j = 1, 6)$. However, for stability in the Lyapunov sense the weaker condition, that all of the roots of $P(s)$ are at least purely imaginary, is required. This condition constrains the motion to a local neighborhood of the nominal stationary solution.

The coefficients of the polynomial $P(s)$ are given by

$$a_0 = 1 \tag{21a}$$

$$a_1 = 0 \tag{21b}$$

$$a_2 = M_{11}^* + M_{22}^* + M_{33}^* + 4 \tag{21c}$$

$$a_3 = 2(M_{21}^* - M_{12}^*) \tag{21d}$$

$$a_4 = M_{11}^* M_{22}^* + M_{11}^* M_{33}^* + M_{22}^* M_{33}^* - M_{23}^* M_{32}^* - M_{13}^* M_{31}^* - M_{12}^* M_{21}^* + 4M_{33}^* \tag{21e}$$

$$a_5 = 2M_{33}^*(M_{21}^* - M_{12}^*) + 2(M_{32}^* M_{13}^* - M_{23}^* M_{31}^*) \tag{21f}$$

$$a_6 = M_{11}^* M_{22}^* M_{33}^* - M_{11}^* M_{23}^* M_{32}^* - M_{33}^* M_{12}^* M_{21}^* - M_{22}^* M_{31}^* M_{13}^* + M_{21}^* M_{32}^* M_{13}^* + M_{12}^* M_{23}^* M_{31}^* \tag{21g}$$

Since $a_1 = 0$, an application of the Routh-Hurwitz criterion implies that at least one eigenvalue will not lie in the left-hand complex plane (i.e., there is at least one eigenvalue with $\gamma_j \geq 0$). Therefore the system does not naturally possess asymptotic stability. Given this fact, the condition for Lyapunov-type stability with purely imaginary eigenvalues, $\gamma_j = 0$ ($j = 1, 6$), will be established. Substituting for $s = i\omega$, the characteristic polynomial $P(s)$ becomes

$$P(j\omega) = -\omega^6 + a_2\omega^4 - ia_3\omega^3 - a_4\omega^2 + ia_5\omega + a_6 \tag{22}$$

For the condition $P(s) = 0$ to hold it is required that both the real and purely imaginary parts of the polynomial are identically zero

$$-\omega^6 + a_2\omega^4 - a_4\omega^2 + a_6 = 0 \tag{23a}$$

$$i\omega(a_5 - \omega^2 a_3) = 0 \tag{23b}$$

Six consistent solutions of Eqs. (23) with $\omega_j^2 > 0$ ($j = 1, 6$) are now required. From Eq. (23b) it is seen that $\omega_1 = 0$, $\omega_{2,3} = \pm\sqrt{(a_5/a_3)}$. However, the solution $\omega_1 = 0$ is obviously inconsistent with Eq. (23a). The remaining solutions $\omega_{2,3}$ are also not generally consistent with Eq. (23a). However, Eq. (23b) is satisfied if $a_3 = a_5 = 0$. The eigenvalues of the system are then determined as conjugate pairs from Eq. (23a), which may or may not have real solutions.

Therefore a necessary, but not sufficient, condition for Lyapunov stability is then $a_3 = 0 \Rightarrow (M_{21}^* - M_{12}^*) = 0$. However, since the potential is conservative $U_{yx} - U_{xy} = 0$, so that $a_3 = 0 \Rightarrow (a_{yx} - a_{xy}) = 0$. Similarly the condition $a_5 = 0$ requires that $a_{zx} - a_{xz} = 0$ and $a_{yz} - a_{zy} = 0$. Taken together these conditions imply that $\beta = 0$ or

$$\nabla \times \boldsymbol{a} = 0 \tag{24}$$

That is, the solar radiation pressure acceleration must be conservative and so must be derivable from some scalar potential. Therefore, the required conditions for Lyapunov-type stability are that $\alpha = 0$ (modified photogravitational system[2]), or $\beta = 0$ (classical restricted system). In practice the solutions away from the Earth will behave as Earth synchronous heliocentric halo orbits, which are found to have regions of Poincaré stability and instability.[3,4]

It has been shown then that the set of new stationary solutions do not possess a natural asymptotic stability and that Lyapunov stability is only possible for the particular solutions when the sail is oriented along the sun line. In general, therefore, a control scheme is required to ensure asymptotic stability. A simple control scheme using proportional and derivative feedback to the sail attitude will now be outlined.

Including first-order variations in the sail attitude $\boldsymbol{n}_0 \rightarrow \boldsymbol{n}_0 + \delta\boldsymbol{n}$, the open-loop variational system becomes

$$\frac{d^2\boldsymbol{\delta}}{dt^2} + \boldsymbol{M}_1 \frac{d\boldsymbol{\delta}}{dt} + \boldsymbol{M}^*\boldsymbol{\delta} = \boldsymbol{N}\delta\boldsymbol{n}, \qquad \boldsymbol{N} = \left.\frac{\partial \boldsymbol{a}}{\partial \boldsymbol{n}}\right|_{r = r_0,\, n = n_0} \tag{25}$$

In the full six-dimensional phase space, with $\boldsymbol{x} = (\boldsymbol{\delta}, d\boldsymbol{\delta}/dt)^T$, the system then becomes

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{P}\boldsymbol{x} + \boldsymbol{Q}\delta\boldsymbol{n}, \quad \boldsymbol{P} = \left\{\begin{array}{c:c} \boldsymbol{0} & \boldsymbol{I} \\ \hdashline -\boldsymbol{M}^* & -\boldsymbol{M}_1 \end{array}\right\}, \quad \boldsymbol{Q} = \left\{\begin{array}{c} \boldsymbol{0} \\ \boldsymbol{N} \end{array}\right\} \tag{26}$$

To proceed further it is necessary to determine if the system is fully controllable. Therefore, the 6×6 controllability matrix $\boldsymbol{C} = \{\boldsymbol{Q}, \boldsymbol{PQ}, \boldsymbol{P}^2\boldsymbol{Q}, \boldsymbol{P}^3\boldsymbol{Q}\}$ must have full rank

$$\boldsymbol{C} = \left\{\begin{array}{c:c:c:c} \boldsymbol{0} & \boldsymbol{N} & -\boldsymbol{M}_1\boldsymbol{N} & -\boldsymbol{M}^*\boldsymbol{N} + \boldsymbol{M}_1^2\boldsymbol{N} \\ \boldsymbol{N} & -\boldsymbol{M}_1\boldsymbol{N} & -\boldsymbol{M}^*\boldsymbol{N} + \boldsymbol{M}_1^2\boldsymbol{N} & 2\boldsymbol{M}^*\boldsymbol{M}_1\boldsymbol{N} - \boldsymbol{M}_1^3\boldsymbol{N} \end{array}\right\} \tag{27}$$

Since $\boldsymbol{N} \neq 0$ if $\boldsymbol{r}_1 \cdot \boldsymbol{n} \neq 0$, in general all the rows of $\boldsymbol{C}$ are linearly independent so that $r(\boldsymbol{C}) = 6$ and the system is fully controllable. The control will then be defined as

$$\delta\boldsymbol{n} = \boldsymbol{\Lambda}_1 \cdot \boldsymbol{\delta} + \boldsymbol{\Lambda}_2 \cdot \left\{\frac{d\boldsymbol{\delta}}{dt}\right\} \tag{28}$$

so that the sail attitude trim is given as a function of the sail position and velocity relative to the nominal stationary solution. The closed-loop system is then given by

$$\frac{d^2\boldsymbol{\delta}}{dt^2} + (\boldsymbol{M}_1 - \boldsymbol{N}\boldsymbol{\Lambda}_2)\frac{d\boldsymbol{\delta}}{dt} + (\boldsymbol{M}^* - \boldsymbol{N}\boldsymbol{\Lambda}_1)\,\boldsymbol{\delta} = 0 \tag{29}$$

Therefore, the feedback control now allows $\gamma_j < 0$ ($j = 1, 6$) with a suitable choice of gain matrices $\boldsymbol{\Lambda}_{1,2}$ so that the stationary solutions may have asymptotic stability. In general the choice of gains will depend to a large extent on operational requirements. (In the far Earth region the control for heliocentric halo-type orbits may be used.[4]) However, to demonstrate the existence of asymptotic stability the gain matrices will be chosen as

$$\boldsymbol{\Lambda}_1 = \boldsymbol{N}^{-1}(\boldsymbol{M}^* - \lambda_2\boldsymbol{I}), \qquad \boldsymbol{\Lambda}_2 = \boldsymbol{N}^{-1}(\boldsymbol{M}_1 - \lambda_1\boldsymbol{I}) \tag{30}$$

Substituting these gains into the variational system, a damped harmonic equation is obtained with the damping proportional to the gain constants $\lambda_{1,2}$

$$\left\{\frac{d^2\boldsymbol{\delta}}{dt^2}\right\} + \lambda_1\left\{\frac{d\boldsymbol{\delta}}{dt}\right\} + \lambda_2\boldsymbol{\delta} = 0 \tag{31}$$

which has a characteristic polynomial

$$s^2 + \lambda_1 s + \lambda_2 = 0 \tag{32}$$

Since the gain constants $\lambda_{1,2}$ may be arbitrarily chosen, the eigenvalues may be chosen to be in the left-hand complex plane ensuring asymptotic stability. A solution of Eq. (31) is then

$$\boldsymbol{\delta} = \boldsymbol{\delta}_0 e^{st}, \qquad s = -\left\{\frac{\lambda_1}{2}\right\} \pm \left\{\frac{\lambda_1^2}{4} - \lambda_2\right\}^{1/2} \tag{33}$$

so that for asymptotic stability it is required that $\lambda_1 > 0$, $\lambda_2 > \lambda_1^2/4$. It has been shown then that, in principle, the stationary solutions are controllable using a feedback control to the sail attitude and that asymptotic stability can therefore be achieved.

Dynamical Equations for the Earth-Moon System

An idealized perfectly reflecting solar sail will now be considered in a corotating reference frame of constant angular velocity $\boldsymbol{\Omega}$ with a point mass Earth m_1 and moon m_2. The dynamics of the Earth-moon restricted three-body system are quite different from the Earth-sun system in that the sun line $\boldsymbol{S}$ is not fixed in the corotating frame but rotates once per synodic lunar month. It will be assumed that the solar radiation pressure is constant in magnitude over the scale of the problem. In the units of the system the Earth-moon distance is taken to be unity. Therefore, the sail loading parameter is now defined as the solar radiation pressure acceleration made dimensionless with respect to the Earth's gravitational acceleration at the lunar distance. The spacecraft mass per unit area is then related to the sail loading parameter by the relation $\sigma = 3.385\beta^{-1}$ gm^{-2}.

The vector dynamical equation for a solar sail in this corotating frame may be written as

$$\frac{d^2\boldsymbol{r}}{dt^2} + 2\boldsymbol{\Omega} \times \frac{d\boldsymbol{r}}{dt} + \nabla U(\boldsymbol{r}) = \boldsymbol{a} \tag{34}$$

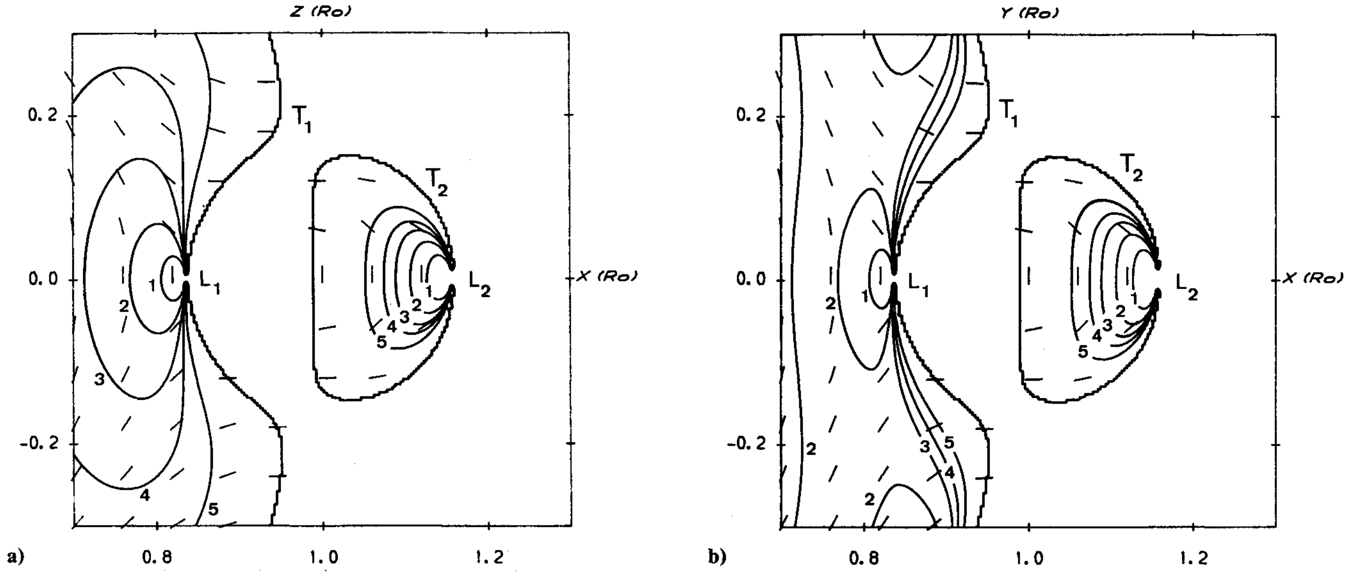

Fig. 4 Section of the level surfaces in the near lunar region a) normal to the plane of the system and b) in the plane of the system at time $\Omega_* t = 0$ deg. Sail loading values are given by: 1) 0.3, 2) 0.6, 3) 1.0, 4) 1.5, and 5) 3.0.

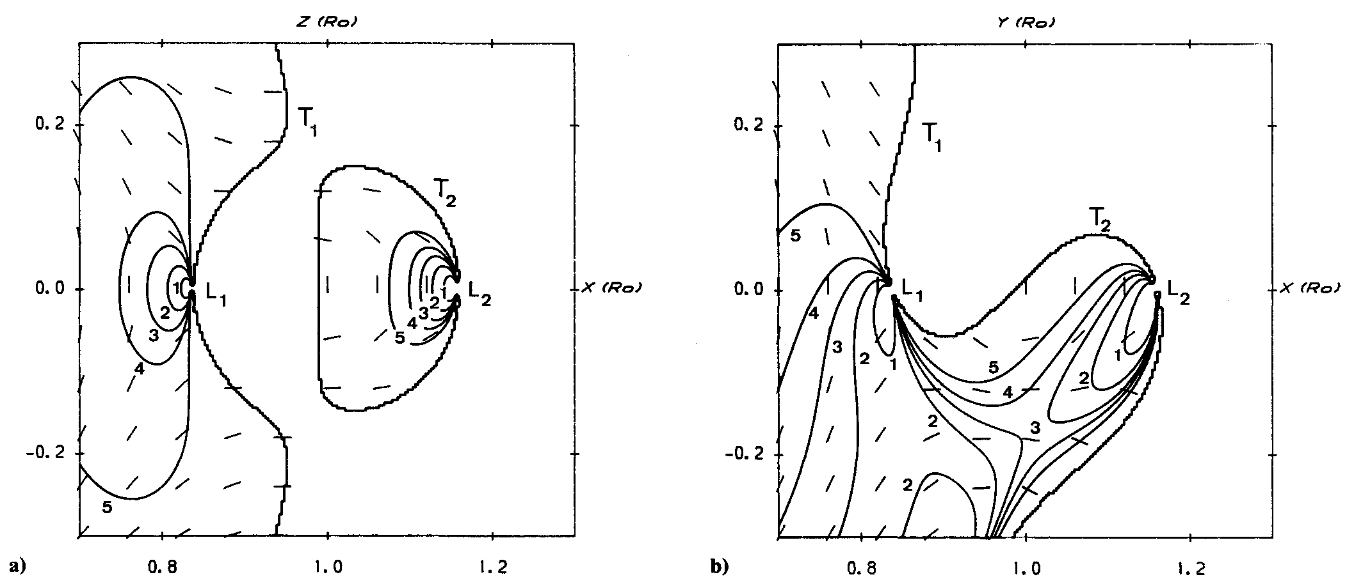

Fig. 5 Section of the level surfaces in the near lunar region a) normal to the plane of the system and b) in the plane of the system at time $\Omega_* t = 45$ deg. Sail loading values are given by: 1) 0.3, 2) 0.6, 3) 1.0, 4) 1.5, and 5) 3.0.

where the corotating three-body potential $U(\boldsymbol{r})$ and the solar radiation pressure acceleration $\boldsymbol{a}$ are given by

$$U(\boldsymbol{r}) = -\left\{ \tfrac{1}{2} |\boldsymbol{\Omega} \times \boldsymbol{r}|^2 + \frac{1-\mu}{|\boldsymbol{r}_1|} + \frac{\mu}{|\boldsymbol{r}_2|} \right\}, \quad \boldsymbol{a} = \beta(\boldsymbol{S} \cdot \boldsymbol{n})^2 \boldsymbol{n} \quad (35)$$

where $\mu = (m_2/m_1 + m_2) = 0.01215$ is the mass ratio of the Earth-moon system. The sail attitude is constrained such that $\boldsymbol{S} \cdot \boldsymbol{n} \geq 0$ and the direction of the sun line is given by

$$\boldsymbol{S} = [\cos(\Omega_* t), \ -\sin(\Omega_* t), \ 0] \quad (36)$$

where $\Omega_* = 0.9252$ is the angular rate of the sun line in the corotating frame in dimensionless units. The small annual changes (± 5 deg) in the inclination of the sun line with respect to the plane of the system are ignored.

By again requiring stationary solutions in the corotating frame and taking vector products, Eq. (34) may be solved to obtain the required sail attitude as

$$\boldsymbol{n} = \frac{\nabla U(\boldsymbol{r})}{|\nabla U(\boldsymbol{r})|} \quad (37)$$

which is time independent. The required sail loading may also be obtained, but is, however, time dependent due to the rotating sun line

$$\beta(t) = \frac{\nabla U(\boldsymbol{r}) \cdot \boldsymbol{n}}{[\boldsymbol{S}(t) \cdot \boldsymbol{n}]^2} \quad (38)$$

As before, the region of existence of stationary solutions is bounded, the boundary being defined by the time-dependent condition $\boldsymbol{S} \cdot \boldsymbol{n} \geq 0$. This condition yields a function $T(\boldsymbol{r}; t) = 0$ defining the time-dependent boundary surface to the regions of existence of solutions. This function is given by

$$T(\boldsymbol{r}; t) = \frac{\partial U(\boldsymbol{r})}{\partial x} \cos(\Omega_* t) - \frac{\partial U(\boldsymbol{r})}{\partial y} \sin(\Omega_* t) \quad (39)$$

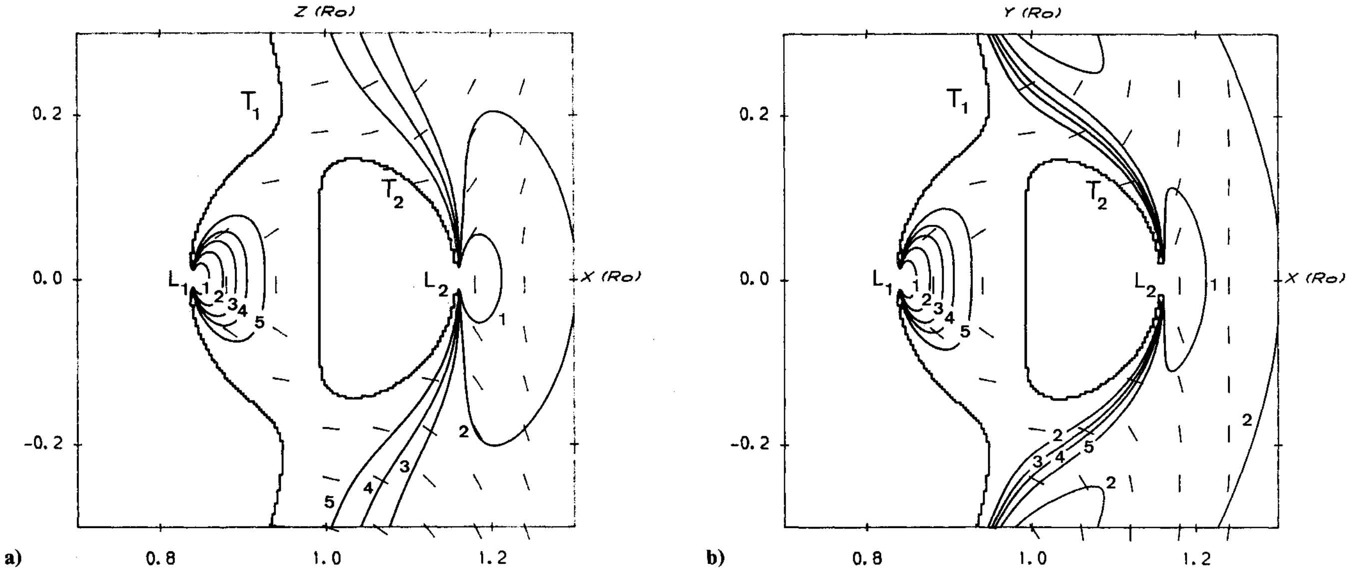

Fig. 6 Section of the level surfaces in the near lunar region a) normal to the plane of the system and b) in the plane of the system at time $\Omega_* t = 180$ deg. Sail loading values are given by: 1) 0.3, 2) 0.6, 3) 1.0, 4) 1.5, and 5) 3.0.

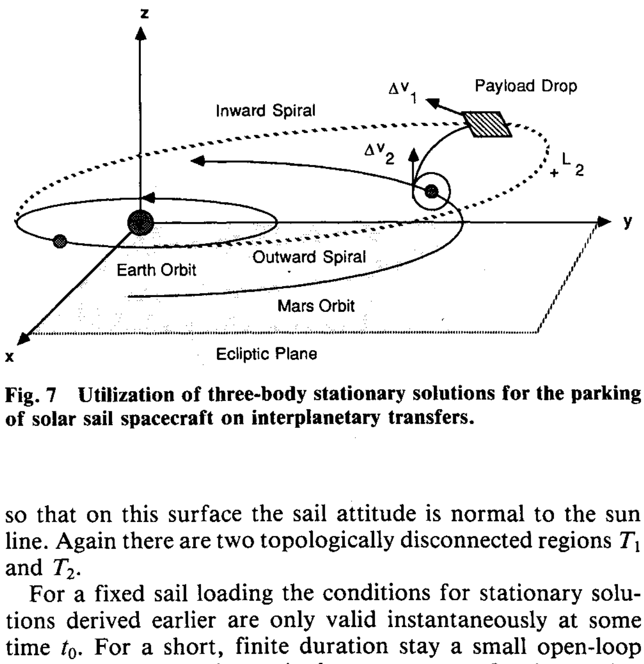

Fig. 7 Utilization of three-body stationary solutions for the parking of solar sail spacecraft on interplanetary transfers.

so that on this surface the sail attitude is normal to the sun line. Again there are two topologically disconnected regions T_1 and T_2.

For a fixed sail loading the conditions for stationary solutions derived earlier are only valid instantaneously at some time t_0. For a short, finite duration stay a small open-loop control acceleration is required to compensate for the moving sun line. This acceleration will require a small variation in the sail loading since the sail attitude required is time independent. Although Eq. (38) gives the required change in sail loading as a function of the sun line position, an approximate expression can be obtained by expanding the solar radiation pressure acceleration about the stationary solution at position r_0 and time t_0. At a time Δt later the condition for continued equilibrium is given by

$$\nabla U(r_0) = a(t_0) + \sum_{j=1}^{\infty} \frac{1}{j!} \frac{\mathrm{d}^j a}{\mathrm{d}t^j} \Delta t^j \tag{40}$$

Therefore, for a short duration stay ($\Delta t \ll 1$) the first-order trim in the sail loading is given by

$$\frac{\Delta\beta}{\beta} = -2S \cdot n \left\{ n \cdot \frac{\mathrm{d}S}{\mathrm{d}t} \right\} \Delta t \tag{41}$$

If $\Delta\beta/\beta$ remains small by limiting Δt to 5×10^{-3}, for example, a stay of 3.3 h is possible. For solar sails with a large trim capability, much longer duration stays would, however, be possible. Furthermore, if $n = S$ then $n \cdot (\mathrm{d}S/\mathrm{d}t) = 0$ so that, to first order, no variation in the sail loading is required.

Stationary Solutions in the Earth-Moon System

Using Eq. (38), level surfaces of constant sail loading in the Earth-moon system may be generated. Again, sections of the surfaces through the x-y and x-z planes are used. Then, only the pitch angle α is required to describe the required sail attitude.

At time $\Omega_* t = 0$ deg the sun line is directed along the Earth-moon line. Sections of the level surfaces of constant sail loading at this time are shown in Fig. 4. The surfaces expand with increasing sail loading in a similar manner to the Earth-sun system. This result is to be expected due to the configuration of the Earth-moon-sun system at this time. Sections of the boundary surfaces T_1 and T_2 are also shown.

Some time later in the synodic month when $\Omega_* t = 45$ deg the topology of the surfaces radically transforms (see Fig. 5). Normal to the plane of the system the surfaces are still symmetric. However, in the plane of the system the surfaces about the L_1 and L_2 points are asymmetric and connect at a loading value of approximately 1.2. Similarly, the boundary surfaces T_1 and T_2 are now connected. It can, however, be seen that the required sail attitude is time independent. Finally, at time $\Omega_* t = 180$ deg the regions of existence of solutions have reversed with respect to $\Omega_* t = 0$ deg, and so solutions are now forbidden within surface T_2 and are allowed outside of surface T_1 (see Fig. 6).

Applications

For interplanetary solar sail missions, a large fraction of the total transfer time is contained in the initial and final planetocentric spirals to and from low planetary orbit and escape. This element of the total transfer time can, however, be eliminated by parking the solar sail near the planetary L_2 point on a level surface of constant sail loading equal to the total spacecraft loading. At the end of an interplanetary trajectory the spacecraft would transfer its payload to a low planetary orbit with impulses from a chemical motor with space storable propellants (see Fig. 7).

The solar sail parking scheme would also allow the sail to be optimally designed for interplanetary space. Planetocentric spiral trajectories require large attitude turning rates, a requirement that puts demands on the sail structure and attitude control mechanisms. Also, for sample returns, if the sample was transferred to the parked sail by chemical means the long spiral out of the planet's gravity well could be avoided. After

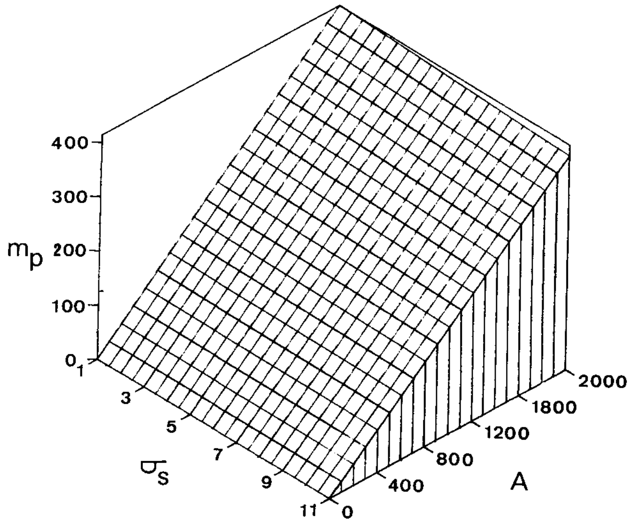

Fig. 8 Isometric projection of a surface of payload mass m_p as a function of sail area A and sail mass per unit area σ_s.

the initial payload drop to the planetary surface, the sail would have an increased loading and so would be stationed on a level surface closer to the planet to await the sample return from the planetary surface. Of course, there is a mass penalty imposed due to the need for propellant to be transported to the destination. However, the shortened total mission duration may more than compensate.

At the Earth-sun L_1 point a number of spacecraft, such as ISEE-3, have been, or will be, positioned for solar observations and for "upwind" observations of the solar wind to be combined with data from near Earth orbiting spacecraft. However, spacecraft positioned directly at the L_1 point would appear in the center of the solar radio disk when observed from Earth and so would be unable to make data returns. This problem is overcome, however, by forcing the spacecraft to execute a ballistic Lagrange point halo orbit. This highly unstable trajectory is a periodic orbit normal to the ecliptic plane so that, when viewed from Earth, the spacecraft appears to orbit around the solar radio disk allowing data returns. To maintain the Lagrange point orbit, however, requires regular stationkeeping maneuvers so that the on-board propellant mass ultimately determines the mission lifetime.

Using a relatively small sail a payload could be positioned at the L_1 point, but simply displaced above the ecliptic plane to avoid the solar radio disk. At a distance of 1.2×10^5 km above the ecliptic plane and 7.2×10^4 km sunward of the L_1 point, a minimized total spacecraft mass per unit area of 205 gm^{-2} is required. For a low performance sail of 5 gm^{-2} and a 10^3 kg payload, only a small 70×70 m square sail is required. A tradeoff between payload mass, sail area, and the sail total mass per unit area is shown in Fig. 8. For low payload masses extremely small sails are required. Since stationkeeping may be performed using trims on the sail attitude, no propellant is required and the mission lifetime is limited only by the lifetime of the payload and the sail. This factor is also of importance for potential payloads as chemical exhaust plumes may corrupt spectral data.

Other space science missions utilizing the three-body solar sail stationary solutions would require a solar sail to be stationed near the L_2 point to investigate the far regions of the geomagnetic tail. Since the sail would be stationary, temporal and spatial variations in the observations may be deconvolved. The full three-dimensional structure of the tail may then be investigated by transferring from in-plane to out-of-plane stationary points. Finally, applications for communications purposes have been made as an extension of the "statite" concept to large geocentric distances where a full three-body analysis is required.[5]

Conclusions

It has been demonstrated that circular restricted three-body systems with solar sail spacecraft have a continuum of new artificial stationary solutions. For the Earth-sun system these new solutions appear as level surfaces of constants sail loading around the classical Lagrange points. A linear stability analysis shows that these solutions are in general unstable, apart from the solutions with the sail oriented along the sun line. However, since it is found that the system is controllable, asymptotic stability may be ensured through the use of feedback control to the sail attitude. These new solutions appear to have interesting potential applications for space science missions.

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